Shorted Operators and Generalized Inverses of Matrices

Sujit Kumar Mitra

Indiana University
Bloomington, Indiana
and
Indian Statistical Institute
New Delhi, India

and

Madan L. Puri

Indiana University
Bloomington, Indiana

Submitted by P. Lax

1. INTRODUCTION

The shorted operator in finite dimensional linear spaces was defined by Anderson [1] in terms of an explicit matrix construction. It was shown to be the solution to a maximization problem. In a subsequent paper by Anderson and Trapp [3], the same maximization problem was used to furnish the definition, and the definition extends itself to an arbitrary Hilbert space (not necessarily finite dimensional).[1] The concepts of a minimum seminorm g-inverse and a semi-least-squares inverse of a complex matrix were introduced and studied by Rao and Mitra [8]. In the present paper we obtain two explicit representations for the shorted operator (in finite dimensional linear spaces) in terms of these inverses. These representations lead to some new theorems on the shorted operators and to new proofs of old theorems. The shorting operation could also be viewed as a hermitian order preserving map from $\mathcal{C}\times\mathcal{C}$ onto $\mathcal{C}$, where $\mathcal{C}$ is the cone of nonnegative definite matrices and $\mathcal{C}\times\mathcal{C}$ is the cartesian product of $\mathcal{C}$ with itself. The operation is not commutative but it satisfies associativity.

*Work supported by the Air Force Office of Scientific Research, AFSC USAF, Grant No. AFOSR 76-2927, and National Science Foundation Grant No. MCS 76-00951. Reproduction in whole or in part permitted for any purpose of the United States Government.

[1]The referee has very kindly pointed out that this alternative definition is actually due to Krein [4].

0024-3795/79/030045+12$01.75

We use the following notation and definitions. For a positive integer n, $\mathcal{E}^n$ is the linear space of complex n-tuples. Column vector representations of vectors in $\mathcal{E}^n$ will be denoted by lower case letters such as y, v, etc. $\mathcal{C}^{m\times n}$ represents the linear space of complex matrices of order $m\times n$. $\mathcal{C}_n$ represents the cone of nonnegative definite (n.n.d.) matrices in $\mathcal{C}^{n\times n}$. Matrices are denoted by capital letters such as A, B, M, etc. For a matrix A, $\mathcal{M}(A)$ denotes its column span, $\mathcal{N}(A)$ its null space and A^* its complex conjugate transpose. If $B,C\in\mathcal{C}_n$, we write $B\geqslant C$ if $B-C\in\mathcal{C}_n$.

DEFINITION. If $\mathcal{S}$ is a subspace of $\mathcal{E}^n$ and $B\in\mathcal{C}_n$, the shorted matrix $\mathcal{S}(B)$ is the unique matrix in $\mathcal{C}_n$ such that

$$\mathcal{M}[\mathcal{S}(B)]\subset\mathcal{S}, \tag{1.1i}$$

$$B\geqslant\mathcal{S}(B), \tag{1.1ii}$$

$$\text{if } C\in\mathcal{C}_n, B\geqslant C \quad\text{and}\quad \mathcal{M}(C)\subset\mathcal{S}, \tag{1.1iii}$$

then $\mathcal{S}(B)\geqslant C$. The existence of $\mathcal{S}(B)$ was established by Anderson and Trapp [3].

For a complex matrix $A\in\mathcal{C}^{m\times n}$, the matrix $G\in\mathcal{C}^{n\times m}$ is a g-inverse of A if $x=Gy$ is a solution of the consistent equation $Ax=y$ $\forall y\in\mathcal{M}(A)$. We represent such an inverse by A^- and the entire class by $\{A^-\}$. G is a g-inverse of A iff

$$AGA=A. \tag{1.2}$$

For a vector $x\in\mathcal{E}^n$ we consider the seminorm of x defined by

$$\|x\|_n=(x^*Nx)^{1/2},$$

where N is a given matrix in $\mathcal{C}_n$. G is a minimum N seminorm g-inverse of A if $G\in\{A^-\}$ and $\forall y\in\mathcal{M}(A)$, $Au=y\Rightarrow$

$$\|Gy\|_n\leqslant\|u\|_n.$$

We represent such a G by $A^-_{m(N)}$ and the class by $\{A^-_{m(N)}\}$. $G\in\{A^-_{m(N)}\}$ iff

$$AGA=A, \qquad (NGA)^*=NGA. \tag{1.3}$$

Let the seminorm of vector $y \in \mathcal{E}^m$ be similarly defined in terms of a given $M \in \mathcal{C}_m$. $x_0 \in \mathcal{E}^n$ is called an M-semi-least-squares solution of the possibly inconsistent equation $Ax = y$ if $\forall x \in \mathcal{E}^n$

$$\|Ax_0 - y\|_m \leqslant \|Ax - y\|_m. \tag{1.4}$$

G is an M-semi-least-squares inverse of A if $\forall y \in \mathcal{E}^m$, Gy is an M-semi-least-squares solution of $Ax = y$. We represent such a G by $A_{l(M)}$ and the class by $\{A_{l(M)}\}$. $G \in \{A_{l(M)}\}$ iff

$$A^*MAG = A^*M, \tag{1.5}$$

or equivalently

$$MAGA = MA, \qquad (MAG)^* = MAG. \tag{1.6}$$

$x_0 \in \mathcal{E}^n$ is a minimum N seminorm M semi-least-squares solution of $Ax = y$ if $\forall x \in \mathcal{E}^n$, either

$$\|Ax_0 - y\|_m < \|Ax - y\|_m \tag{1.7}$$

or

$$\|Ax_0 - y\|_m = \|Ax - y\|_m \quad \text{and} \quad \|x_0\|_n \leqslant \|x\|_n. \tag{1.8}$$

G is a minimum N seminorm M-semi-least-squares inverse of A if $\forall y \in \mathcal{E}^m$, Gy is a minimum N seminorm M-semi-least-squares solution of $Ax = y$. We represent such a G by A_{MN}, the class by $\{A_{MN}\}$. $G \in \{A_{MN}\}$ iff

$$A^*MAG = A^*M, \qquad \mathfrak{M}(NG) \subset \mathfrak{M}(A^*MA), \tag{1.9}$$

or equivalently

$$\begin{aligned} MAGA &= MA, \qquad (MAG)^* = MAG; \\ NGAC &= NG, \qquad (NGA)^* = NGA; \\ &\mathfrak{M}(NG) \subset \mathfrak{M}(A^*MA). \end{aligned} \tag{1.10}$$

The results on minimum seminorm and semi-least-squares inverses listed in the section are proved in [8], and those on minimum seminorm semileast squares inverse in [7].

2. THE SHORTED OPERATOR IN TERMS OF A MINIMUM SEMINORM g-INVERSE

We show that the matrix NGA as considered in (1.3) is indeed the shorted operator $\mathcal{S}(N)$, where $\mathcal{S}=\mathfrak{M}(A^*)$. Before we do this let us first establish the uniqueness and nonnegativity of NGA.

THEOREM 2.1. *If* $G\in\{A^-_{m(N)}\}$,

(a) *NGA is unique with respect to choice of G in this class;*
(b) $NGA\in\mathcal{C}_n$, $(N-NGA)\in\mathcal{C}_n$;
(c) *$\mathfrak{M}(N-NGA)$ is virtually disjoint with $\mathfrak{M}(A^*)$;*
(d) *NGA and $N-NGA$ are disjoint matrices* [5], *or equivalently*

$$\mathfrak{M}(N) = \mathfrak{M}(NGA)\oplus\mathfrak{M}(N-NGA);$$

(e) $\mathfrak{M}(NGA)=\mathfrak{M}(N)\cap\mathfrak{M}(A^*)$.

Proof.

(a): If $G_1,G_2\in\{A^-_{m(N)}\}$,

$$[(G_1-G_2)A]^*N(G_1-G_2)A = N(G_1-G_2)A(G_1-G_2)A = 0.$$

Since $N\in\mathcal{C}_n$, this implies $N(G_1-G_2)A=0$.

(b): Since $NGA=(GA)^*NGA$, $N-NGA=(I-GA)^*N(I-GA)$ and $N\in\mathcal{C}_n$, (b) is established.

(c): Let $A^*u=(N-NGA)v$ be a vector in $\mathfrak{M}(A^*)\cap\mathfrak{M}(N-NGA)$, then $A^*u=A^*G^*A^*u=A^*G^*(N-NGA)v=0$. This proves (c).

(d): follows from (c), since $NGA=A^*G^*N\Rightarrow\mathfrak{M}(NGA)\subset\mathfrak{M}(A^*)$.

(e): (d) implies that if $x\in\mathfrak{M}(N)\cap\mathfrak{M}(A^*)$, it can be uniquely expressed as $x=x_1+x_2$ such that $x_1\in\mathfrak{M}(NGA)$, $x_2\in\mathfrak{M}(N-NGA)$. However, $x_2=x-x_1\in\mathfrak{M}(A^*)$. Using (c), therefore, we have $x_2=0$ and $x=x_1\in\mathfrak{M}(NGA)$. This shows $\mathfrak{M}(N)\cap\mathfrak{M}(A^*)\subset\mathfrak{M}(NGA)$. That $\mathfrak{M}(NGA)\subset\mathfrak{M}(N)\cap\mathfrak{M}(A^*)$ is trivial. This concludes the proof of (e) and of Theorem 2.1. ■

The unique matrix $NA^-_{m(N)}A$ will henceforth be denoted by $I(N,A)$. $P_0=A^*(A^-_{m(N)})^*$ is a projector onto $\mathfrak{M}(A^*)$. If $P=A^*G^*$ is also a projector onto $\mathfrak{M}(A^*)$, the minimum seminorm property of $A^-_{m(N)}$ implies

$$P_0NP_0^* \leqslant PNP^*.$$

Let $\mathcal{P}$ be the class of all matrices which are projectors onto $\mathfrak{M}(A^*)$. We have the following theorem:[2]

THEOREM 2.2.

$$I(N,A) = \min_{P\in\mathcal{P}} PNP^*.$$

The following theorem identifies $I(N,A)$ as a shorted operator.

THEOREM 2.3. *If* $\mathcal{S} = \mathfrak{M}(A^*)$, $I(N,A)$ *is the shorted operator* $\mathcal{S}(N)$.

Proof. We recall the definition (1.1). The condition of (1.1i) follows from Theorem 2.1(e), (1.1ii) from Theorem 2.1(b). Also, if $C\in\mathcal{C}_n$, $\mathfrak{M}(C)\subset\mathcal{S}$ and $N\geqslant C$, then

$$\begin{aligned}x^*I(N,A)x &= x^*A^*(A^-_{m(N)})^*NA^-_{m\,(N)}Ax\\ &\geqslant x^*A^*(A^-_{m(N)})^*CA^-_{m(N)}Ax = x^*Cx.\end{aligned}$$

Thus the condition (1.1iii) is also satisfied. This completes proof of Theorem 2.3. ■

THEOREM 2.4.

(a) *If* P_N *is the orthogonal projector onto* $\mathfrak{M}(N)$ *under the inner product* $(x,z)=z^*x$, *then* $I(P_N,A)$ *is the orthogonal projector onto* $\mathfrak{M}(N)\cap\mathfrak{M}(A^*)$.

(b) *Every g-inverse of N is a g-inverse of* $I(N,A)$.

(c)

$$\{I(N,A)\}^+ = I(P_N,A)N^-I(P_N,A).$$

(d) *If x is an eigenvector of N corresponding to the eigenvalue* λ *and* $x\in\mathfrak{M}(A^*)$, *then x is also an eigenvector of* $I(N,A)$ *corresponding to the same eigenvalue* λ.

Proof.

(a): is given as Corollary 6 in [3].

(b): is a consequence of the fact that

$$\begin{aligned}I(N,A)N^-I(N,A) &= A^*(A^-_{m(N)})^*NN^-NA^-_{m(N)}A\\ &= A^*(A^-_{m(N)})^*NA^-_{m(N)}A = I(N,A).\end{aligned}$$

[2]Since in Theorem 2.3, $I(N,A)$ is shown to be the shorted operator $\mathcal{S}(N)$, Theorem 2.2 is same as Theorem 5 of [3]. However, the approaches are different.

(c): follows from (a), (b) and Note 8, p. 52 of Rao and Mitra [8].
(d):

$$\begin{aligned} Nx = \lambda x \quad \Rightarrow \quad I(N,A)x &= A^*(A^-_{m(N)})^* Nx \\ &= \lambda A^*(A^-_{m(N)})^* x \\ &= \lambda x \qquad \text{if} \quad x \in \mathcal{M}(A^*). \end{aligned}$$

Note that the condition $x \in \mathcal{M}(A^*)$ is superfluous if $\lambda = 0$. This establishes (d), and the proof of Theorem 2.4 is concluded. ■

NOTE 1. A general solution to a g-inverse of $I(N,A)$ is given by $N^- + X$ where N^- is an arbitrary g-inverse of N and X is an arbitrary solution of the equation $AXA^* = 0$. Since $N^- \in \{I(N,A)^-\}$, a general solution to $I(N,A)^-$ is given by $N^- + X_0$, where X_0 is an arbitrary solution to the equation $I(N,A)X_0 I(N,A) = 0$. In view of Theorem 2.1(b) and (c), any such X_0 can always be expressed as $X_0 = X_1 + X_2$, where $NX_1N = 0$ and $AX_2A^* = 0$. This establishes the above claim.

The following two theorems are concerned with square matrices.

THEOREM 2.5. *If NA is hermitian,*

$$\mathcal{M}[N - I(N,A)] = \mathcal{M}(N) \cap \mathcal{N}(A^*).$$

Proof.

$$NA = A^*N \quad \Rightarrow \quad A^*[N - NA^-_{m(N)}A] = NA - NAA^-_{m(N)}A = 0.$$

Dimensionality considerations show

$$\mathcal{M}[N - I(N,A)] = \mathcal{M}(N) \cap \mathcal{N}(A^*).$$ ■

THEOREM 2.6. *If $A, N \in \mathcal{C}_n$, then*

(a) *$P[I(N,A), I(A,N)] = P(N,A)$, where $P(\ ,0)$ represents parallel sum* [2, 6, 8];

(b) *$[I(A,N)]^-$ and $[I(N,A)]^-$ are arbitrary g-inverses, then*

$$[I(A,N)]^- + [I(N,A)]^- = N^- + A^-$$

for suitable choice of N^- and A^-.

Proof.

(a):

$$P(N,A) = N(N+A)^{-}A = I(N,A)(N+A)^{-}A + [N-I(N,A)](N+A)^{-}A$$

$$= I(N,A)[I(N,A)+A]^{-}A$$

$$= P[I(N,A),A],$$

noting that since $N-I(N,A)$ and $I(N,A)+A$ are disjoint matrices by Lemma 2.7 of Mitra [5], every g-inverse of $N+A$ is a g-inverse of $I(N,A)+A$. Also $[N-I(N,A)](N+A)^{-}[I(N,A)+A]=0 \Rightarrow [N-I(N,A)](N+A)^{-}A=0$. A similar argument shows $P[I(N,A),A]=P[I(N,A),I(A,N)]$.

(b) follows from Theorem 10.18(d) of Rao and Mitra [8] or alternatively from Note 1 following Theorem 2.4.

3. THE SHORTED OPERATOR IN TERMS OF SEMI-LEAST-SQUARES INVERSE

We obtain results here which are similar to those in Sec. 2.

THEOREM 3.1. *If* $G \in \{A_{l(M)}\}$,

(a) *MAG is unique with respect to choice of G in this class,*

(b) $MAG \in \mathcal{C}_m$, $M - MAG \in \mathcal{C}_m$,

(c) $\mathfrak{M}(MAG)$ *is virtually disjoint with* $\mathfrak{N}(A^*)$

(d) *MAG and* $M-MAG$ *are disjoint matrices, or equivalently*

$$\mathfrak{M}(M) = \mathfrak{M}(MAG) \oplus \mathfrak{M}(M-MAG),$$

(e) $\mathfrak{M}(M-MAG)=\mathfrak{M}(M)\cap\mathfrak{N}(A^*)$.

Proof.

(a): If $G_1, G_2 \in \{A_{l(M)}\}$,

$$[A(G_1-G_2)]^*MA(G_1-G_2) = (G_1-G_2)^*A^*MA(G_1-G_2)$$

$$= (G_1-G_2)^*(A^*M-A^*M) = 0.$$

Since $M \in \mathcal{C}_m$, this implies $MAG_1 = MAG_2$.

(b): Since $MAG = G^*A^*MAG$, $M - MAG = (I - AG)^*M(I - AG)$ and $M \in \mathcal{C}_m$, (b) is established.

(c): Let $MAGu$ be a vector in $\mathfrak{M}(MAG) \cap \mathfrak{N}(A^*)$: then

$$A^*Mu = A^*MAGu = 0$$
$$\Rightarrow \quad MAGu = G^*A^*Mu = 0.$$

This proves (c).

(d): $A^*(M - MAG) = 0 \Rightarrow \mathfrak{M}(M - MAG) \subset \mathfrak{N}(A^*)$. Hence (d) follows from (c).

(e): Clearly $\mathfrak{M}(M - MAG) \subset \mathfrak{M}(M) \cap \mathfrak{N}(A^*)$. Let $y \in \mathfrak{M}(M) \cap \mathfrak{N}(A^*)$. Using (d), we write $y = y_1 + y_2$ such that $y_1 \in \mathfrak{M}(MAG)$, $y_2 \in \mathfrak{M}(M - MAG)$. Observe that $y_1 = y - y_2 \in \mathfrak{N}(A^*)$. In view of (c), this implies $y_1 = 0$, $y = y_2 \in \mathfrak{M}(M - MAG)$. Thus $\mathfrak{M}(M) \cap \mathfrak{N}(A^*) \subset \mathfrak{M}(M - MAG)$ and (e) is established. ■

The unique matrix $M - MAA_{l(M)}$ will henceforth be denoted by $J(M,A)$. $Q_0 = AA_{l(M)}$ is a matrix in $\mathcal{C}^{m\times m}$ with column span contained in $\mathfrak{M}(A)$. If $Q = AG$ is any other such matrix, the semi-least-squares property of $A_{l(M)}$ implies

$$(I - Q_0)^*M(I - Q_0) \leqslant (I - Q)^*M(I - Q).$$

Let $\mathfrak{Q}$ be the class of all matrices of all matrices in $\mathcal{C}^{m\times m}$ with column span contained in $\mathfrak{M}(A)$. We have thus the following theorem.

THEOREM 3.2.

$$J(M,A) = \min_{Q \in \mathfrak{Q}} (I - Q)^*M(I - Q).$$

THEOREM 3.3. *If* $\mathcal{S} = \mathfrak{N}(A^*)$, *then* $J(M,A)$ *is the shorted operator* $\mathcal{S}(M)$.

Proof. Observe that the condition (1.1i) follows from Theorem 3.1(e), and (1.1ii) from Theorem 3.1(b). Also if $C \in \mathcal{C}_m$, $\mathcal{C}(C) \subset \mathcal{S}$ and $M \geqslant C$, then

$$y^*J(M,A)y = y^*(I - AA_{l(M)})^*M(I - AA_{l(M)})y$$
$$\geqslant y^*(I - AA_{l(M)})^*C(I - AA_{l(M)})y$$
$$= y^*Cy.$$

Thus the condition (1.1iii) is also satisfied, and the proof of Theorem 3.3 is concluded. ■

THEOREM 3.4.

(a) *If* P_M *is the orthogonal projector onto* $\mathfrak{M}(M)$ *under the inner product* $(y,w)=w^*y$, *then* $J(P_M,A)$ *is the orthogonal projector onto* $\mathfrak{M}(M)\cap\mathfrak{N}(A^*)$.

(b) *Every g-inverse of* M *is a g-inverse of* $J(M,A)$.

(c) $\{J(M,A)\}^+=J(P_M,A)M^-J(P_M,A)$.

(d) *If* y *is an eigenvector of* M *corresponding to the eigenvalue* λ *and* $y\in\mathfrak{N}(A^*)$, *then* y *is an eigenvector of* $J(M,A)$ *corresponding to the same eigenvalue.*

Proof. The proof is similar to that of Theorem 2.4 and is therefore omitted. ■

NOTE 2. A general solution to a g-inverse of $J(M,A)$ is given by $M^-+AY+W^*A^*$ where M^- is an arbitrary g-inverse of M, and $Y,W\in\mathcal{C}^{n\times m}$ but are otherwise arbitrary. This claim is established in the same way as Note 1 following Theorem 2.4.

The next two theorems are concerned with square matrices.

THEOREM 3.5. *If* MA *is hermitian,*

$$\mathfrak{M}[J(M,A)] = \mathfrak{M}(M)\cap\mathfrak{M}(A^*).$$

Proof. The proof is trivial and is therefore omitted. ■

THEOREM 3.6. *If* $A,M\in\mathcal{C}_m$,

(a) $J(M,A)J(A,M)=J(M,A)I(M,A)=J(M,A)I(A,M)= P[J(M,A),J(A,M)]=P[J(M,A),I(M,A)]=P[J(M,A),I(A,M)]=0$

(b) $[M-J(M,A)][A-J(A,M)]=MA$.

4. THE MINIMUM SEMINORM SEMI-LEAST-SQUARES INVERSE

Theorem 4.1 gives an alternative definition of generalized Moore-Penrose inverses (Mitra and Rao [7]).

THEOREM 4.1. *The following statements are equivalent*:

(a) $G \in \{A_{MN}\}$,
(b) $M - MAG = J(M,A)$, $NGA = I(N,A^*MA)$, $\operatorname{rank} NGA = \operatorname{rank} NG$.

Proof.

(a) $\Rightarrow$ (b): Since $\{A_{MN}\} \subset \{A_{l(M)}\}$, the first part of (b) follows from the definition of $J(M,A)$. Also (1.10)$\Rightarrow$

$$NGA = A^*G^*NGA,$$

$$N - NGA = (I - GA)^*N(I - GA).$$

Hence $NGA \in \mathcal{C}_n$ and $N \geqslant NGA$. Clearly $\mathfrak{M}(NGA) \subset \mathfrak{M}(A^*MA)$. Also if $C \in \mathcal{C}_n$, $N \geqslant C$ and $\mathfrak{M}(C) \subset \mathfrak{M}(A^*MA)$,

$$x^*NGAx = x^*A^*G^*NGAx \geqslant x^*A^*G^*CGAx = x^*Cx,$$

since $A^*MAGA = A^*MA$. Hence NGA satisfies the conditions (1.1i), ii, iii) of (1.1), and it is seen that $NGA = I(N,A^*MA)$. The last part of (b) is trivial.

(b) $\Rightarrow$ (a): From Theorem 3.1(e) it is seen that $M - MAG = J(M,A) \Rightarrow G \in \{A_{l(M)}\}$. Also $NGA = I(N,A^*MA) \Rightarrow \mathfrak{M}(NGA) \subset \mathfrak{M}(A^*MA)$, and the rank condition implies $\mathfrak{M}(NG) = \mathfrak{M}(NGA) \subset \mathfrak{M}(A^*MA)$. Thus G satisfies (1.9) and hence $G \in \{A_{MN}\}$. ∎

It was noted in Mitra and Rao [7] that the conditions

$$A^*MAG = A^*M, \qquad G^*NGA = G^*N \tag{4.1}$$

do not determine a minimum N seminorm M-semi-least-squares inverse of A. A matrix G which satisfies (4.1) is denoted by the symbol $A_{(MN)}$. The following theorem gives an explicit description of the class $\{A_{(MN)}\}$.

THEOREM 4.2. $G \in \{A_{(MN)}\}$ *iff* $G \in \{A_{M_0N}\}$ *for some* $M_0 = M + M_1$, *where* $M_1 \in \mathcal{C}_m$ *and is such that* A^*MA *and* A^*M_1A *are disjoint matrices (that is, the row spans of* A^*MA *and* A^*M_1A *are virtually disjoint and so are the column spans).*

Proof.

"*If*" *part*: A^*MA and A^*M_1A are disjoint
$\Leftrightarrow \operatorname{rank}(A^*M_0A) = \operatorname{rank}(A^*MA) + \operatorname{rank}(A^*M_1A)$ (by Lemma 2.1 of Mitra [5])
$\Leftrightarrow \operatorname{rank}(A^*M_0) = \operatorname{rank}(A^*M) + \operatorname{rank}(A^*M_1)$

$\Leftrightarrow A^*M$ and A^*M_1 are disjoint matrices. Hence $G \in \{A_{M_0N}\}$
$\Rightarrow A^*M_0AG = A^*M_0,\ G^*NGA = G^*N$
$\Rightarrow A^*MAG + A^*M_1AG = A^*M + A^*M_1,\ G^*NGA = G^*N$
$\Rightarrow A^*MAG = A^*M,\ G^*NGA = G^*N$
$\Rightarrow G \in \{A_{(MN)}\}$.

"*Only if*" *part*: Conversely, let $G \in \{A_{(MN)}\}$. Put $A^*M_1A = NGA - I(NGA, A^*MA)$, $A^*M_2A = I(NGA, A^*MA)$ and note, using Theorem 2.1(d), that A^*M_1A and A^*M_2A are disjoint matrices. Hence

$$NGAG = NG \quad \Rightarrow \quad A^*M_1AG = A^*M_1.$$

Further, by construction, A^*M_1A is disjoint with A^*MA. Hence $G \in \{A_{(MN)}\}$
$\Leftrightarrow A^*MAG = A^*M,\ G^*NGA = G^*N$
$\Rightarrow A^*(M+M_1)AG = A^*(M+M_1),\ G^*NGA = G^*N$
$\Rightarrow G \in \{A_{M_0N}\}$,since

$$\mathfrak{M}(NG) = \mathfrak{M}(NGA) = \mathfrak{M}(A^*M_1A) \oplus \mathfrak{M}(A^*M_2A)$$
$$\subset \mathfrak{M}(A^*M_1A) + \mathfrak{M}(A^*MA) = \mathfrak{M}(A^*M_0A). \qquad \blacksquare$$

THEOREM 4.3. $\{A_{(MN)}\} = \{A_{MN}\}$ *iff*

$$\mathfrak{M}(N) \cap \mathfrak{M}(A^*) = \mathfrak{M}(N) \cap \mathfrak{M}(A^*MA). \tag{4.2}$$

Proof. The "if" part is trivial. For the "only if" part observe that if (4.2) is untrue, $\mathfrak{M}(N) \cap \mathfrak{M}(A^*MA)$ is a proper subset of $\mathfrak{M}(N) \cap \mathfrak{M}(A^*)$. Consider the decomposition

$$I(N,A) = A^*M_1A + A^*M_2A$$

as in the proof of the "only if" part of Theorem 4.2, where $A^*M_2A = I[I(N,A), A^*MA]$. Put $M_0 = M + M_1$, and observe that $G = A_{M_0N} \in \{A_{(MN)}\}$. However,

$$\mathfrak{M}(NGA) = \mathfrak{M}(N) \cap \mathfrak{M}(A^*M_0A) = \mathfrak{M}(N) \cap \mathfrak{M}[I(N,A)]$$
$$= \mathfrak{M}(N) \cap \mathfrak{M}(A^*) \not\subset \mathfrak{M}(A^*MA).$$

Hence $G \notin \{A_{MN}\}$. Thus negation of (4.2) implies negation of (1.9). This completes the proof of Theorem 4.3. $\blacksquare$

The authors would like to express their sincere thanks to the referee for his prompt and thorough reading of the manuscript and many valuable comments and suggestions.

REFERENCES

1 W. N. Anderson, Jr., Shorted operators, *SIAM J. Appl. Math.* 20:520–525 (1971).
2 W. N. Anderson, Jr. and R. J. Duffin, Series and parallel addition of matrices, *J. Math. Anal. Appl.* 11:576–594 (1963).
3 W. N. Anderson, Jr. and G. Trapp, Shorted operators II, *SIAM J. Appl. Math.* 28:60–71 (1975).
4 M. G. Krein, The theory of self-adjoint extensions of semi-bounded Hermitian operators and its applications, *Mat. Sbornik N.S.* 20(62):431–495 and 21(63):365–404 (1947).
5 S. K. Mitra, Fixed rank solutions of linear matrix equations, *Sankhyā*, Ser. A, 35:387–392 (1972).
6 S. K. Mitra and Madan L. Puri, On parallel sum and difference of matrices, *J. Math. Anal. Appl.* 44:92–97 (1973).
7 S. K. Mitra and C. R. Rao, Projections under seminorms and generalized Moore-Penrose inverses, *Linear Algebra Appl.* 9:155–167 (1974).
8 C. R. Rao and S. K. Mitra, *Generalized Inverse of Matrices and Its Applications*, Wiley, New York, 1971.

Received 9 March 1978